ISRAEL JOURNAL OF MATHEMATICS, Vol. 72, Nos. 1–2, 1990

ON HOPF ALGEBRAS AND RIGID MONOIDAL CATEGORIES

BY

K.-H. ULBRICH†

Institute of Mathematics, University of Tsukuba, Tsukuba-shi, Ibaraki 305, Japan

ABSTRACT

Let $\mathscr{C}$ be a neutral Tannakian category over a field k. By a theorem of Saavedra Rivano there exists a commutative Hopf algebra A over k such that $\mathscr{C}$ is equivalent to the category of finite dimensional right A-comodules. We review Saavedra Rivano's construction of the bialgebra A and show that A has still an antipode if the symmetry condition on the monoidal structure of $\mathscr{C}$ is removed.

Introduction

Let k be a field, $\mathscr{C}$ a k-linear, abelian category which is essentially small, and let $\omega: \mathscr{C} \to \mathscr{V}ec_f(k)$ be a k-linear, exact and faithful functor from $\mathscr{C}$ into the category of finite dimensional k-vector spaces. By [3], p. 136, 2.6.3, there exists a k-coalgebra A such that ω factors through a k-linear equivalence $\mathscr{C} \to \mathscr{C}omod_f(A)$ of $\mathscr{C}$ with the category of finite dimensional right A-comodules. If in addition $\mathscr{C}$ has a rigid, symmetric monoidal structure (i.e. a symmetric monoidal structure such that every object has a dual object), and if ω is a symmetric monoidal functor, then A is a commutative Hopf algebra; it represents the functor

$$(1) \qquad T \mapsto \mathrm{Aut}^{\otimes}(\omega \otimes T)$$

which associates to a commutative k-algebra T the group of monoidal natural automorphisms of $\omega \otimes T: \mathscr{C} \to \mathscr{M}od(T)$, $X \mapsto \omega(X) \otimes T$, see [2], thm. 2.11, [3], II, 4.1.

† Supported by a grant from JSPS.

Current address: Département de Mathématiques, Université Paris-Nord, Villetaneuse, France.

Received July 23, 1989

Suppose now that we drop from the above assumptions the symmetry condition on the monoidal structure of $\mathscr{C}$. Then A is still a bialgebra, which however does no longer represent the functor (1), because A may be non-commutative.

The aim of this note is to show that A is in fact still a Hopf algebra by deriving the existence of an antipode of A from the dual object functor of $\mathscr{C}$.

1. We first collect some facts on dual objects in monoidal categories, [1], [3]. Let $\mathscr{C}$ be a monoidal category, with product $\otimes$ and neutral object k. Let $X \in \mathscr{C}$. An object X^* of $\mathscr{C}$ is said to be a (left) dual of X if there exist morphisms

$$ev: X^* \otimes X \to k, \qquad \pi: k \to X \otimes X^*,$$

satisfying

$$id = (1 \otimes ev)(\pi \otimes 1): X \to X \otimes X^* \otimes X \to X,$$

$$id = (ev \otimes 1)(1 \otimes \pi): X^* \to X^* \otimes X \otimes X^* \to X^*.$$

Given such morphisms ev and π, it is easy to see that for all $Y, Z \in \mathscr{C}$ the maps

$$\mathrm{Hom}(Z \otimes X, Y) \to \mathrm{Hom}(Z, Y \otimes X^*), \qquad f \mapsto (f \otimes 1)(1 \otimes \pi),$$

$$\mathrm{Hom}(Z, Y \otimes X^*) \to \mathrm{Hom}(Z \otimes X, Y), \qquad g \mapsto (1 \otimes ev)(g \otimes 1)$$

are inverse to each other. Hence the functor $\mathscr{C} \to \mathscr{C}$, $Y \mapsto Y \otimes X^*$, is a right adjoint of $\mathscr{C} \to \mathscr{C}$, $Z \mapsto Z \otimes X$. Taking $Y = k$ shows X^* is uniquely determined up to isomorphism. More precisely, suppose that $(\overline{X}^*, \overline{ev}, \overline{\pi})$ is another dual of X. Define

$$\tau = (ev \otimes 1)(1 \otimes \overline{\pi}): X^* \to \overline{X}^*.$$

Then τ is an isomorphism and

$$ev = \overline{ev}(\tau \otimes 1), \qquad \overline{\pi} = (1 \otimes \tau)\pi. \tag{2}$$

Suppose now that $\mathscr{C}$ is rigid monoidal, i.e. every object $X \in \mathscr{C}$ has a dual X^*. Then for any morphism $f: X \to Y$ in $\mathscr{C}$ we can define

$$f^* = (ev \otimes 1)(1 \otimes f \otimes 1)(1 \otimes \pi): Y^* \to X^*.$$

This gives rise to a contravariant function $\mathscr{C} \to \mathscr{C}$, $X \mapsto X^*$, which is called the dual object functor of $\mathscr{C}$. Let $\omega: \mathscr{C} \to \mathscr{D}$ be a monoidal functor from $\mathscr{C}$ into a rigid monoidal category $\mathscr{D}$. Then $\omega(X^*)$ is naturally a dual object for $\omega(X)$. The corresponding τ is given by

$$\tau_X = (ev \otimes 1)(1 \otimes \omega(\pi)): \omega(X)^* \to \omega(X^*);$$

τ_X is natural in X and, in particular, satisfies (2).

EXAMPLE. Let k be a commutative ring, $\mathscr{C} = \mathscr{M}od(k)$, and $\otimes = \bigotimes_k$. Then $P \in \mathscr{C}$ has a dual object if and only if P is finitely generated and projective, [1], prop. 2.6. It follows that the same holds for $\mathscr{C} = \mathscr{C}omod(A)$, the category of right A-comodules for a Hopf k-algebra A. In fact, if $V \in \mathscr{C}omod(A)$ is finitely generated and projective over k, then $V^* = \mathrm{Hom}_k(V, k)$ has a right A-comodule structure $V^* \to V^* \otimes A \cong \mathrm{Hom}_k(V, A)$ defined by

$$\left(\sum g_{(0)} \otimes g_{(1)}\right)(x) = \sum g(x_{(0)})\lambda(x_{(1)}), \qquad x \in V, \quad g \in V^*,$$

where $\lambda: A \to A$ is the antipode of A. The canonical maps ev: $V^* \otimes V \to k$ and $\pi: k \to V \otimes V^*$, $1 \mapsto$ projective coordinate system, are then A-collinear. If $\lambda^2 = id$, then also $V \xrightarrow{\sim} V^{**}$ is A-collinear.

2. In the following let $\mathscr{C} = (\mathscr{C}, \otimes, \mathscr{k})$ be a monoidal category, which is essentially small (i.e. equivalent to a small category). The bialgebra A mentioned in the introduction can be obtained more generally from any monoidal functor

$$\omega: \mathscr{C} \to \mathscr{P}(k)$$

from $\mathscr{C}$ into the category of finitely generated, projective modules over a commutative ring k. It may be defined as follows, cf. [3], p. 101. For any k-module M let $\omega \otimes M$ denote the functor $\mathscr{C} \to \mathscr{M}od(k)$, $X \mapsto \omega(X) \otimes M$, and $\mathrm{Hom}(\omega, \omega \otimes M)$ the k-module of all natural transformations $\omega \to \omega \otimes M$. It follows from [3], p. 99, 1.3.2.1, that the functor

$$\mathscr{M}od(k) \to \mathscr{E}ns, \qquad M \mapsto \mathrm{Hom}(\omega, \omega \otimes M),$$

is representable. Choose a representing k-module A and natural isomorphisms

$$\rho_M: \mathrm{Hom}_k(A, M) \xrightarrow{\sim} \mathrm{Hom}(\omega, \omega \otimes M), \qquad M \in \mathscr{M}od(k).$$

Set $\alpha = \rho_A(id_A): \omega \to \omega \otimes A$, $d = (\alpha \otimes 1)\alpha: \omega \to \omega \otimes A \otimes A$, and let $e: \omega \to \omega \otimes k$ be the canonical isomorphism. Then the coalgebra structure $\delta: A \to A \otimes A$ and $\varepsilon: A \to k$ is given by

$$\rho_{A \otimes A}(\delta) = d, \qquad \rho_k(\varepsilon) = e,$$

and one obtains a functor $\mathscr{C} \to \mathscr{C}omod(A)$, $X \mapsto (\omega(X), \alpha_X)$.

For the algebra structure of A, consider the functor

$$\omega^2\colon \mathscr{C} \times \mathscr{C} \to \mathscr{P}(k), \qquad (X, Y) \mapsto \omega(X) \otimes \omega(Y).$$

It follows from [3], p. 100, 1.3.3.1, that the map

$$(3) \qquad \rho_M^2\colon \mathrm{Hom}_k(A \otimes A, M) \to \mathrm{Hom}(\omega^2, \omega^2 \otimes M),$$

defined by $\rho_M^2(\mu)(x \otimes y) = \Sigma\, x_{(0)} \otimes y_{(0)} \otimes \mu(x_{(1)} \otimes y_{(1)})$ for all $x \in \omega(X)$, $y \in \omega(Y)$, is an isomorphism. Define homomorphisms $m_{X,Y}$ for $X, Y \in \mathscr{C}$ by

$$m_{X,Y}\colon \omega(X) \otimes \omega(Y) \cong \omega(X \otimes Y) \xrightarrow{\alpha} \omega(X \otimes Y) \otimes A \cong \omega(X) \otimes \omega(Y) \otimes A.$$

This gives an element $m \in \mathrm{Hom}(\omega^2, \omega^2 \otimes A)$ and the multiplication $\mu\colon A \otimes A \to A$ is the preimage of m under (3). Finally, the unit map $\iota\colon k \to A$ is defined by

$$\iota\colon k \cong \omega(\not{k}) \xrightarrow{\alpha} \omega(\not{k}) \otimes A \cong A.$$

REMARK 1. Let $\phi_A\colon \mathscr{C}omod(A) \to \mathscr{M}od(k)$ be the forgetful functor. Obviously, the functor $\Omega\colon \mathscr{C} \to \mathscr{C}omod(A)$ defined above satisfies $\omega = \phi_A \Omega$ as monoidal functors $\mathscr{C} \to \mathscr{M}od(k)$. If (A', Ω') is any other such pair, then $\Omega' = \mathscr{C}omod(\phi)\Omega$ for a uniquely determined bialgebra map $\phi\colon A \to A'$.

THEOREM. *Suppose that $\mathscr{C}$ is rigid monoidal. Then the k-bialgebra A defined above is a Hopf algebra.*

PROOF. For any k-module M and $X \in \mathscr{C}$ we have a canonical isomorphism

$$(4) \qquad \mathrm{Hom}_k(\omega(X), \omega(X) \otimes M) \xrightarrow{\sim} \mathrm{Hom}_k(\omega(X)^*, \omega(X)^* \otimes M)$$

since $\omega(X)$ is finitely generated and projective. Furthermore, by Section 1 we have natural isomorphisms

$$\tau_X\colon \omega(X)^* \to \omega(X^*), \qquad X \in \mathscr{C},$$

compatible with ev and π, (2). For $\nu \in \mathrm{Hom}(\omega, \omega \otimes M)$ define $\tilde{\nu}_X$ to be the preimage under (4) of the composite

$$(\tau_X^{-1} \otimes 1)\nu_{X^*}\tau_X\colon \omega(X)^* \to \omega(X)^* \otimes M.$$

This yields a map $l_M\colon \mathrm{Hom}(\omega, \omega \otimes M) \to \mathrm{Hom}(\omega, \omega \otimes M)$, $\nu \mapsto \tilde{\nu}$, which is natural in M, and hence corresponds to a k-linear map $\lambda\colon A \to A$ satisfying $l_M \rho_M = \rho_M \mathrm{Hom}(\lambda, 1)$. We claim that λ is an antipode of A, i.e. $\mu(\lambda \otimes 1)\delta = \iota\varepsilon = \mu(1 \otimes \lambda)\delta$. Since $\mathrm{Hom}_k(A, A) \cong \mathrm{Hom}(\omega, \omega \otimes A)$, it is enough to show

$$\sum x_{(0)} \otimes \lambda(x_{(1)}) x_{(2)} = x \otimes 1 = \sum x_{(0)} \otimes x_{(1)} \lambda(x_{(2)}) \tag{5}$$

for all $x \in \omega(X)$, $X \in \mathscr{C}$. For the left-hand equation first observe that the following diagram is commutative:

$$\begin{array}{ccc} \omega(X)^* \otimes \omega(X) & \xrightarrow{\alpha \otimes \alpha} & \omega(X)^* \otimes A \otimes \omega(X) \otimes A \\ ev \downarrow & & \downarrow 1 \otimes \mu \\ k \xrightarrow{\ \iota\ } A & \xleftarrow{ev \otimes 1} & \omega(X)^* \otimes \omega(X) \otimes A \end{array} \tag{6}$$

as follows from the naturality of α applied to ev: $X^* \otimes X \rightarrow \mathscr{k}$. In (6), by abuse of notation, the map $\alpha: \omega(X)^* \rightarrow \omega(X)^* \otimes A$ denotes the comodule structure induced from $\omega(X^*)$ via τ_X. If we identify $\omega(X)^* \otimes A \cong \mathrm{Hom}_k(\omega(X), A)$, then

$$\alpha(g)(x) = \sum g(x_{(0)}) \lambda(x_{(1)}), \qquad x \in \omega(X), \quad g \in \omega(X)^*.$$

It follows that the canonical isomorphism

$$\mathrm{Hom}_k(\omega(X)^* \otimes \omega(X), A) \xrightarrow{\sim} \mathrm{Hom}_k(\omega(X), \omega(X) \otimes A)$$

transforms (6) into the first equation of (5). However, the same arguments with ev replaced by π: $\mathscr{k} \rightarrow X \otimes X^*$ give also the second equation of (5), and this completes the proof.

REMARK 2. If the dual object functor $\mathscr{C} \rightarrow \mathscr{C}$ is an equivalence, then λ is bijective.

REFERENCES

1. P. Deligne, *Catégories Tannakiennes*, preprint.

2. P. Deligne and J. Milne, *Tannakian Categories*, Lecture Notes in Math. **900**, Springer-Verlag, Berlin, 1982, pp. 101–228.

3. N. Saavedra Rivano, *Catégories Tannakiennes*, Lecture Notes in Math. **265**, Springer-Verlag, Berlin, 1972.